### Sistema

Pared — resorte ($k$) — masa ($m$) — resorte ($k$) — masa ($m$) — resorte ($k$) — Pared.

## Desacoplar ecuaciones

### Ecuaciones de movimiento

$$m\ddot{\psi}_a = -k\psi_a + k(\psi_b - \psi_a) = -2k\psi_a + k\psi_b \tag{1}$$

$$m\ddot{\psi}_b = -k\psi_b + k(\psi_a - \psi_b) = k\psi_a - 2k\psi_b \tag{2}$$

### Desacoplamiento

Definimos las coordenadas normales:

$$\psi_1 = \psi_a + \psi_b, \qquad \psi_2 = \psi_a - \psi_b$$

Sumando (1) + (2):

$$m\ddot{\psi}_1 = -k\psi_1 \quad\Longrightarrow\quad \ddot{\psi}_1 = -\frac{k}{m}\,\psi_1$$

Restando (1) − (2):

$$m\ddot{\psi}_2 = -3k\psi_2 \quad\Longrightarrow\quad \ddot{\psi}_2 = -\frac{3k}{m}\,\psi_2$$

### Ecuaciones desacopladas

$$\ddot{\psi}_1 + \omega_1^2\,\psi_1 = 0, \qquad \ddot{\psi}_2 + \omega_2^2\,\psi_2 = 0$$

### Solución de los modos normales

$$\psi_1(t) = A_1\cos(\omega_1 t + \varphi_1) \tag{3}$$

$$\psi_2(t) = A_2\cos(\omega_2 t + \varphi_2) \tag{4}$$

### Solución para cada masa

Invirtiendo $\psi_1 = \psi_a + \psi_b$, $\psi_2 = \psi_a - \psi_b$:

$$\psi_a(t) = \frac{1}{2}\left[\psi_1(t) + \psi_2(t)\right] = \frac{A_1}{2}\cos(\omega_1 t + \varphi_1) + \frac{A_2}{2}\cos(\omega_2 t + \varphi_2) \tag{5}$$

$$\psi_b(t) = \frac{1}{2}\left[\psi_1(t) - \psi_2(t)\right] = \frac{A_1}{2}\cos(\omega_1 t + \varphi_1) - \frac{A_2}{2}\cos(\omega_2 t + \varphi_2) \tag{6}$$

Las cuatro constantes $A_1$, $A_2$, $\varphi_1$, $\varphi_2$ se determinan con las condiciones iniciales $\psi_a(0)$, $\psi_b(0)$, $\dot{\psi}_a(0)$, $\dot{\psi}_b(0)$.

# Formulación matricial

El sistema de ecuaciones:

$$\ddot{\psi}_a = -\frac{2k}{m}\psi_a + \frac{k}{m}\psi_b \tag{7}$$

$$\ddot{\psi}_b = \frac{k}{m}\psi_a - \frac{2k}{m}\psi_b \tag{8}$$

Se escribe como $\ddot{\mathbf{\Psi}} = -D\,\mathbf{\Psi}$, con:

$$\mathbf{\Psi} = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \qquad D = \frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

## Problema de autovalores

Proponemos $\mathbf{\Psi}(t) = \mathbf{v}\,e^{i\omega t}$, lo que lleva a:

$$D\,\mathbf{v} = \omega^2\,\mathbf{v}$$

$$\det\left(D - \omega^2\mathbb{I}\right) = 0 \quad\Longrightarrow\quad \det\begin{pmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{pmatrix} = 0$$

$$\left(\frac{2k}{m} - \omega^2\right)^2 - \frac{k^2}{m^2} = 0 \quad\Longrightarrow\quad \frac{2k}{m} - \omega^2 = \pm\frac{k}{m}$$

## Autovalores y autovectores

$$\boxed{\omega_1^2 = \frac{k}{m}}, \qquad \boxed{\omega_2^2 = \frac{3k}{m}}$$

Para $\omega_1^2 = \dfrac{k}{m}$: $\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix}\mathbf{v}_1 = 0 \quad\Longrightarrow\quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Para $\omega_2^2 = \dfrac{3k}{m}$: $\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix}\mathbf{v}_2 = 0 \quad\Longrightarrow\quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

## Solución general

$$\mathbf{\Psi}(t) = A_1\,\mathbf{v}_1\cos(\omega_1 t + \varphi_1) + A_2\,\mathbf{v}_2\cos(\omega_2 t + \varphi_2)$$

$$\begin{pmatrix} \psi_a(t) \\ \psi_b(t) \end{pmatrix} = \frac{A_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t + \varphi_1) + \frac{A_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t + \varphi_2)$$

# Dos masitas colgando

## Sistema

Techo — resorte ($k$) — masa ($m$) — resorte ($k$) — masa ($m$). Con $\mathbf{g}$

## Ecuaciones de movimiento

$$\left.\begin{aligned} m\ddot{\psi}_a &= -k\psi_a + k(\psi_b - \psi_a) \\ m\ddot{\psi}_b &= k(\psi_b - \psi_a) \end{aligned}\right\} \implies \begin{cases} \ddot{\psi}_a = -\dfrac{2k}{m}\psi_a + \dfrac{k}{m}\psi_b \\ \ddot{\psi}_b = \dfrac{k}{m}\psi_a - \dfrac{k}{m}\psi_b \end{cases}$$

En forma matricial $\ddot{\boldsymbol{\Psi}} = -D\,\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \qquad D = \begin{pmatrix} 2\frac{k}{m} & -1\frac{k}{m} \\ -1\frac{k}{m} & 1\frac{k}{m} \end{pmatrix}$$

## Problema de autovalores

$$\det\left(D - \omega^2\mathbb{I}\right) = 0 \quad\implies\quad \det\begin{pmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - \omega^2 \end{pmatrix} = 0$$

$$\left(\frac{2k}{m} - \omega^2\right)\left(\frac{k}{m} - \omega^2\right) - \frac{k^2}{m^2} = 0$$

$$\omega^4 - \frac{3k}{m}\omega^2 + \frac{k^2}{m^2} = 0$$

$$\omega^2 = \frac{k}{2m}\left(3 \pm \sqrt{5}\right)$$

## Frecuencias normales

$$\boxed{\omega_1^2 = \frac{k}{2m}\left(3 - \sqrt{5}\right)}, \qquad \boxed{\omega_2^2 = \frac{k}{2m}\left(3 + \sqrt{5}\right)}$$

## Autovectores

Para cada $\omega_n^2$, de la primera fila: $\left(\frac{2k}{m} - \omega_n^2\right)\psi_a = \frac{k}{m}\psi_b$, es decir:

$$\frac{\psi_b}{\psi_a} = \frac{2k/m - \omega_n^2}{k/m}$$

Para $\omega_1^2 = \dfrac{k}{2m}(3 - \sqrt{5})$:

$$\frac{\psi_b}{\psi_a} = \frac{1 + \sqrt{5}}{2} = \varphi \quad\implies\quad \mathbf{v}_1 \propto \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$$

Para $\omega_2^2 = \dfrac{k}{2m}(3 + \sqrt{5})$:

$$\frac{\psi_b}{\psi_a} = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\varphi} \quad\implies\quad \mathbf{v}_2 \propto \begin{pmatrix} 1 \\ -1/\varphi \end{pmatrix}$$

donde $\varphi = \frac{1+\sqrt{5}}{2}$ es el número áureo.

### Solución general

$$\mathbf{\Psi}(t) = A_1\,\mathbf{v}_1\cos(\omega_1 t + \varphi_1) + A_2\,\mathbf{v}_2\cos(\omega_2 t + \varphi_2)$$

# Tres masas acopladas

## Sistema

Masa ($m$) — resorte ($k$) — Masa ($M$) — resorte ($k$) — masa ($m$). Tres masas libres.

## Ecuaciones de movimiento

Sean $\psi_a$, $\psi_b$, $\psi_c$ los desplazamientos respecto del equilibrio de la masa izquierda ($m$), central ($M$) y derecha ($m$):

$$m\ddot{\psi}_a = k(\psi_b - \psi_a) \tag{9}$$
$$M\ddot{\psi}_b = -k(\psi_b - \psi_a) - k(\psi_b - \psi_c) = k\psi_a - 2k\psi_b + k\psi_c \tag{10}$$
$$m\ddot{\psi}_c = -k(\psi_c - \psi_b) \tag{11}$$

$$\ddot{\psi}_a = -\frac{k}{m}\psi_a + \frac{k}{m}\psi_b \tag{12}$$
$$\ddot{\psi}_b = \frac{k}{M}\psi_a - \frac{2k}{M}\psi_b + \frac{k}{M}\psi_c \tag{13}$$
$$\ddot{\psi}_c = \frac{k}{m}\psi_b - \frac{k}{m}\psi_c \tag{14}$$

En forma matricial $\ddot{\mathbf{\Psi}} = -D\,\mathbf{\Psi}$:

$$\mathbf{\Psi} = \begin{pmatrix} \psi_a \\ \psi_b \\ \psi_c \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{M} & \frac{2k}{M} & -\frac{k}{M} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{pmatrix} = \begin{pmatrix} \alpha & -\gamma & 0 \\ -\gamma & \beta & -\gamma \\ 0 & -\gamma & \alpha \end{pmatrix}$$

## Problema de autovalores

$$\det\left(D - \omega^2\mathbb{I}\right) = 0 \quad\Longrightarrow\quad \det\begin{pmatrix} \frac{k}{m} - \omega^2 & -\frac{k}{m} & 0 \\ -\frac{k}{M} & \frac{2k}{M} - \omega^2 & -\frac{k}{M} \\ 0 & -\frac{k}{m} & \frac{k}{m} - \omega^2 \end{pmatrix} = 0$$

Desarrollando por la primera fila:

$$\left(\frac{k}{m} - \omega^2\right)\left[\left(\frac{2k}{M} - \omega^2\right)\left(\frac{k}{m} - \omega^2\right) - \frac{k^2}{mM}\right] - \frac{k}{m}\left[-\frac{k}{M}\left(\frac{k}{m} - \omega^2\right)\right] = 0$$

$$\left(\frac{k}{m} - \omega^2\right)\left[\left(\frac{2k}{M} - \omega^2\right)\left(\frac{k}{m} - \omega^2\right) - \frac{k^2}{mM}\right] + \frac{k^2}{mM}\left(\frac{k}{m} - \omega^2\right) = 0$$

$$\left(\frac{k}{m} - \omega^2\right)\left[\left(\frac{2k}{M} - \omega^2\right)\left(\frac{k}{m} - \omega^2\right)\right] = 0$$

Factorizando:

$$\omega^2\left(\frac{k}{m} - \omega^2\right)\left(\frac{k}{m} + \frac{2k}{M} - \omega^2\right) = 0$$

## Frecuencias normales

$$\boxed{\omega_1^2 = 0}, \qquad \boxed{\omega_2^2 = \frac{k}{m}}, \qquad \boxed{\omega_3^2 = \frac{k}{m} + \frac{2k}{M}}$$

## Autovectores

Para $\omega_1^2 = 0$ (traslación rígida):

$$\mathbf{v}_1 \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Para $\omega_2^2 = \dfrac{k}{m}$ (la masa central no se mueve):

$$\mathbf{v}_2 \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Para $\omega_3^2 = \dfrac{k}{m} + \dfrac{2k}{M}$ (masas externas en fase, central en contrafase):

$$\mathbf{v}_3 \propto \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

## Solución general

$$\mathbf{\Psi}(t) = A_1\,\mathbf{v}_1(C_1 + C_2 t) + A_2\,\mathbf{v}_2\cos(\omega_2 t + \varphi_2) + A_3\,\mathbf{v}_3\cos(\omega_3 t + \varphi_3)$$

El modo $\omega_1 = 0$ corresponde a traslación uniforme del sistema completo (no hay paredes).

# Dos péndulos acoplados

Dos péndulos de masa $m$ y longitud $\ell$, acoplados por un resorte de constante $k$ y longitud natural $\ell_0$ conectado en el extremo inferior. La distancia entre los puntos de suspensión es $d = \ell_0$.

## Ecuaciones de movimiento

$$m\ell^2\ddot{\theta}_a = -mg\ell\sin\theta_a - k(\ell\sin\theta_b - \ell\sin\theta_a + d - \ell_0)\ell\cos\theta_a \tag{15}$$

$$m\ell^2\ddot{\theta}_b = -mg\ell\sin\theta_b + k(\ell\sin\theta_b - \ell\sin\theta_a + d - \ell_0)\ell\cos\theta_b \tag{16}$$

Como $d = \ell_0$, se cancelan:

$$m\ell^2\ddot{\theta}_a = -mg\ell\sin\theta_a - k\ell^2(\sin\theta_b - \sin\theta_a)\cos\theta_a \tag{17}$$

$$m\ell^2\ddot{\theta}_b = -mg\ell\sin\theta_b + k\ell^2(\sin\theta_b - \sin\theta_a)\cos\theta_b \tag{18}$$

Para ángulos pequeños ($\sin\theta \approx \theta$, $\cos\theta \approx 1$), dividiendo por $m\ell^2$:

$$\ddot{\theta}_a = -\frac{g}{\ell}\,\theta_a - \frac{k}{m}(\theta_b - \theta_a) = -\left(\frac{g}{\ell} + \frac{k}{m}\right)\theta_a + \frac{k}{m}\,\theta_b \tag{19}$$

$$\ddot{\theta}_b = -\frac{g}{\ell}\,\theta_b + \frac{k}{m}(\theta_b - \theta_a) = \frac{k}{m}\,\theta_a - \left(\frac{g}{\ell} + \frac{k}{m}\right)\theta_b \tag{20}$$

En forma matricial $\ddot{\boldsymbol{\Psi}} = -D\,\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} = \begin{pmatrix} \theta_a \\ \theta_b \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{g}{\ell} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} \end{pmatrix}$$

## Problema de autovalores

$$\left(\frac{g}{\ell} + \frac{k}{m} - \omega^2\right)^2 - \frac{k^2}{m^2} = 0 \quad\Longrightarrow\quad \frac{g}{\ell} + \frac{k}{m} - \omega^2 = \pm\frac{k}{m}$$

## Frecuencias normales

$$\boxed{\omega_1^2 = \frac{g}{\ell}}, \qquad \boxed{\omega_2^2 = \frac{g}{\ell} + \frac{2k}{m}}$$

## Autovectores

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(en fase, el resorte no se deforma)}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{(en contrafase)}$$

## Solución general

$$\mathbf{\Psi}(t) = A_1\,\mathbf{v}_1 \cos(\omega_1 t + \phi_1) + A_2\,\mathbf{v}_2 \cos(\omega_2 t + \phi_2)$$

## Ejemplo: batido

Con condiciones iniciales $\theta_a(0) = \theta_0$, $\theta_b(0) = 0$, $\dot{\theta}_a(0) = \dot{\theta}_b(0) = 0$:

$$A_1 = A_2 = \frac{\theta_0}{2}, \qquad \phi_1 = \phi_2 = 0$$

$$\theta_a(t) = \frac{\theta_0}{2}[\cos(\omega_1 t) + \cos(\omega_2 t)] = \theta_0 \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_2 + \omega_1}{2} t\right) \tag{21}$$

$$\theta_b(t) = \frac{\theta_0}{2}[\cos(\omega_1 t) - \cos(\omega_2 t)] = \theta_0 \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_2 + \omega_1}{2} t\right) \tag{22}$$

La energía se transfiere periódicamente entre los péndulos con frecuencia de batido $\omega_{\text{bat}} = \omega_2 - \omega_1$.