Condiciones iniciales — m,M,m

Tres partículas de masas m, M, m conectadas por resortes de constante k.

Ecuaciones de movimiento

$$\begin{cases} m\,\ddot{x}_1 = -k(x_1 - x_2) \\ M\,\ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3) \\ m\,\ddot{x}_3 = -k(x_3 - x_2) \end{cases}$$

En forma matricial $\ddot{\mathbf{\Psi}} + \mathbf{D}\,\mathbf{\Psi} = 0$:

$$\mathbf{\Psi} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} \dfrac{k}{m} & -\dfrac{k}{m} & 0 \\ -\dfrac{k}{M} & \dfrac{2k}{M} & -\dfrac{k}{M} \\ 0 & -\dfrac{k}{m} & \dfrac{k}{m} \end{pmatrix}$$

Autovalores y autovectores

Definimos $\alpha = k/m$ y $\gamma = k/M$ para simplificar. El polinomio característico es:

$$\det(\mathbf{D} - \lambda\mathbf{I}) = \det\begin{pmatrix} \alpha-\lambda & -\alpha & 0 \\ -\gamma & 2\gamma-\lambda & -\gamma \\ 0 & -\alpha & \alpha-\lambda \end{pmatrix}$$

Desarrollando por la primera fila:

$$= (\alpha-\lambda)[(2\gamma-\lambda)(\alpha-\lambda) - \gamma\alpha] + \alpha[-\gamma(\alpha-\lambda)]$$

$$= (\alpha-\lambda)[(2\gamma-\lambda)(\alpha-\lambda) - 2\alpha\gamma] = (\alpha-\lambda)\,\lambda[\lambda - (\alpha+2\gamma)] = 0$$

$$\boxed{\omega_1^2 = 0}, \qquad \boxed{\omega_2^2 = \alpha = \frac{k}{m}}, \qquad \boxed{\omega_3^2 = \alpha + 2\gamma = \frac{k}{m} + \frac{2k}{M}}$$

Para los autovectores, resolvemos $(\mathbf{D} - \omega_i^2\mathbf{I})\,\mathbf{v}_i = 0$:

- $\omega_1^2 = 0$: de las filas 1 y 3, $v_1 = v_2 = v_3$.
- $\omega_2^2 = \alpha$: fila 1 da $v_2 = 0$; fila 2 da $v_1 = -v_3$.
- $\omega_3^2 = \alpha + 2\gamma$: fila 1 da $v_2 = -\frac{2\gamma}{\alpha}v_1 = -\frac{2m}{M}v_1$; fila 3 da $v_3 = v_1$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\dfrac{2m}{M} \\ 1 \end{pmatrix}$$

El modo $\omega_1^2 = 0$ corresponde a la traslación uniforme del centro de masa.

Solución general

$$\mathbf{\Psi}(t) = (A_1 + B_1\,t)\,\mathbf{v}_1 + A_2\,\mathbf{v}_2\cos(\omega_2 t + \varphi_2) + A_3\,\mathbf{v}_3\cos(\omega_3 t + \varphi_3)$$

Componente a componente:

$$\begin{cases} \psi_1(t) = (A_1 + B_1 t) + A_2\cos(\omega_2 t + \varphi_2) + A_3\cos(\omega_3 t + \varphi_3) \\ \psi_2(t) = (A_1 + B_1 t) - \dfrac{2m}{M}\,A_3\cos(\omega_3 t + \varphi_3) \\ \psi_3(t) = (A_1 + B_1 t) - A_2\cos(\omega_2 t + \varphi_2) + A_3\cos(\omega_3 t + \varphi_3) \end{cases}$$

Velocidades:

$$\begin{cases} \dot\psi_1(t) = B_1 - A_2\,\omega_2\sin(\omega_2 t + \varphi_2) - A_3\,\omega_3\sin(\omega_3 t + \varphi_3) \\ \dot\psi_2(t) = B_1 + \dfrac{2m}{M}\,A_3\,\omega_3\sin(\omega_3 t + \varphi_3) \\ \dot\psi_3(t) = B_1 + A_2\,\omega_2\sin(\omega_2 t + \varphi_2) - A_3\,\omega_3\sin(\omega_3 t + \varphi_3) \end{cases}$$

CI: $\psi_1(0) = \psi_0$, $\psi_3(0) = -\psi_0$, velocidades nulas

Condiciones iniciales: $\psi_1(0) = \psi_0$, $\psi_2(0) = 0$, $\psi_3(0) = -\psi_0$, $\dot\psi_i(0) = 0$.
Evaluando las velocidades en $t = 0$:

$$\begin{cases} B_1 - A_2\,\omega_2\sin\varphi_2 - A_3\,\omega_3\sin\varphi_3 = 0 \\ B_1 + \dfrac{2m}{M}\,A_3\,\omega_3\sin\varphi_3 = 0 \\ B_1 + A_2\,\omega_2\sin\varphi_2 - A_3\,\omega_3\sin\varphi_3 = 0 \end{cases}$$

Restando (i) − (iii):

$$-2A_2\,\omega_2\sin\varphi_2 = 0 \quad\Longrightarrow\quad \boxed{\varphi_2 = 0}$$

Sumando (i) + (iii):

$$2B_1 - 2A_3\,\omega_3\sin\varphi_3 = 0 \quad\Longrightarrow\quad B_1 = A_3\,\omega_3\sin\varphi_3$$

De (ii):

$$B_1 = -\frac{2m}{M}\,A_3\,\omega_3\sin\varphi_3$$

Igualando: $\left(1 + \dfrac{2m}{M}\right)A_3\,\omega_3\sin\varphi_3 = 0$, luego $A_3\sin\varphi_3 = 0$ y $\boxed{B_1 = 0}$.
Evaluando las posiciones en $t = 0$ (con $\varphi_2 = 0$):

$$\begin{cases} A_1 + A_2 + A_3\cos\varphi_3 = \psi_0 \\ A_1 - \dfrac{2m}{M}\,A_3\cos\varphi_3 = 0 \\ A_1 - A_2 + A_3\cos\varphi_3 = -\psi_0 \end{cases}$$

Restando (i) − (iii):

$$2A_2 = 2\psi_0 \quad\Longrightarrow\quad \boxed{A_2 = \psi_0}$$

Sumando (i) + (iii):

$$2A_1 + 2A_3\cos\varphi_3 = 0 \quad\Longrightarrow\quad A_1 = -A_3\cos\varphi_3$$

De (ii): $A_1 = \dfrac{2m}{M}\,A_3\cos\varphi_3$. Igualando: $\left(1 + \dfrac{2m}{M}\right)A_3\cos\varphi_3 = 0$, luego $A_3\cos\varphi_3 = 0$.

Junto con $A_3 \sin\varphi_3 = 0$: $\boxed{A_3 = 0}$, y $\boxed{A_1 = 0}$.

$$\boxed{\boldsymbol{\Psi}(t) = \psi_0\,\mathbf{v}_2\cos(\omega_2 t)} \qquad \Longrightarrow \qquad \begin{cases} \psi_1(t) = \psi_0\cos(\omega_2 t) \\ \psi_2(t) = 0 \\ \psi_3(t) = -\psi_0\cos(\omega_2 t) \end{cases}$$

Se excita únicamente el modo 2.

CI: $\psi_1(0) = \psi_0$, $\psi_3(0) = -\psi_0$, $\dot\psi_i(0) = \dot\psi_0$

Condiciones iniciales: $\psi_1(0) = \psi_0$, $\psi_2(0) = 0$, $\psi_3(0) = -\psi_0$, $\dot\psi_i(0) = \dot\psi_0$.
Evaluando las velocidades en $t = 0$:

$$\begin{cases} B_1 - A_2\,\omega_2\sin\varphi_2 - A_3\,\omega_3\sin\varphi_3 = \dot\psi_0 \\ B_1 + \dfrac{2m}{M}\,A_3\,\omega_3\sin\varphi_3 = \dot\psi_0 \\ B_1 + A_2\,\omega_2\sin\varphi_2 - A_3\,\omega_3\sin\varphi_3 = \dot\psi_0 \end{cases}$$

Restando (i) − (iii):

$$-2A_2\,\omega_2\sin\varphi_2 = 0 \quad \Longrightarrow \quad \boxed{\varphi_2 = 0}$$

Sumando (i) + (iii):

$$2B_1 - 2A_3\,\omega_3\sin\varphi_3 = 2\dot\psi_0 \quad \Longrightarrow \quad B_1 = \dot\psi_0 + A_3\,\omega_3\sin\varphi_3$$

De (ii):

$$B_1 = \dot\psi_0 - \frac{2m}{M}\,A_3\,\omega_3\sin\varphi_3$$

Igualando: $\left(1 + \dfrac{2m}{M}\right)A_3\,\omega_3\sin\varphi_3 = 0$, luego $A_3\sin\varphi_3 = 0$ y $\boxed{B_1 = \dot\psi_0}$.
Las condiciones de posición son idénticas al caso anterior: $\boxed{A_2 = \psi_0}$, $\boxed{A_3 = 0}$, $\boxed{A_1 = 0}$.

$$\boxed{\boldsymbol{\Psi}(t) = \dot\psi_0\,t\,\mathbf{v}_1 + \psi_0\,\mathbf{v}_2\cos(\omega_2 t)} \qquad \Longrightarrow \qquad \begin{cases} \psi_1(t) = \dot\psi_0\,t + \psi_0\cos(\omega_2 t) \\ \psi_2(t) = \dot\psi_0\,t \\ \psi_3(t) = \dot\psi_0\,t - \psi_0\cos(\omega_2 t) \end{cases}$$

Se excitan los modos 1 y 2.

Dos péndulos acoplados

Dos péndulos de masa m y longitud ℓ, acoplados por un resorte de constante k y longitud natural ℓ_0 conectado en el extremo inferior. La distancia entre los puntos de suspensión es $d = \ell_0$.

Ecuaciones de movimiento

$$m\ell\ddot\theta_a = -mg\ell\sin\theta_a - k(\ell\sin\theta_b - \ell\sin\theta_a + d - \ell_0)$$
$$m\ell\ddot\theta_b = -mg\ell\sin\theta_b + k(\ell\sin\theta_b - \ell\sin\theta_a + d - \ell_0)$$

Como $d = \ell_0$, se cancelan:

$$\begin{aligned} m\ell\ddot{\theta}_a &= -mg\ell\sin\theta_a - k\ell(\sin\theta_b - \sin\theta_a) \\ m\ell\ddot{\theta}_b &= -mg\ell\sin\theta_b + k\ell(\sin\theta_b - \sin\theta_a) \end{aligned}$$

Para ángulos pequeños ($\sin\theta \approx \theta$), dividiendo por $m\ell$:

$$\begin{aligned} \ddot{\theta}_a &= -\frac{g}{\ell}\,\theta_a - \frac{k}{m}(\theta_b - \theta_a) = -\left(\frac{g}{\ell} + \frac{k}{m}\right)\theta_a + \frac{k}{m}\,\theta_b \\ \ddot{\theta}_b &= -\frac{g}{\ell}\,\theta_b + \frac{k}{m}(\theta_b - \theta_a) = \frac{k}{m}\,\theta_a - \left(\frac{g}{\ell} + \frac{k}{m}\right)\theta_b \end{aligned}$$

En forma matricial $\ddot{\mathbf{\Psi}} + \mathbf{D}\,\mathbf{\Psi} = 0$:

$$\mathbf{\Psi} = \begin{pmatrix} \theta_a \\ \theta_b \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} \frac{g}{\ell} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} \end{pmatrix}$$

Problema de autovalores

$$\left(\frac{g}{\ell} + \frac{k}{m} - \omega^2\right)^2 - \frac{k^2}{m^2} = 0 \quad \Longrightarrow \quad \frac{g}{\ell} + \frac{k}{m} - \omega^2 = \pm\frac{k}{m}$$

Frecuencias normales y autovectores:

$$\boxed{\omega_1^2 = \frac{g}{\ell}}, \qquad \boxed{\omega_2^2 = \frac{g}{\ell} + \frac{2k}{m}}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solución general

$$\mathbf{\Psi}(t) = A_1\,\mathbf{v}_1\cos(\omega_1 t + \phi_1) + A_2\,\mathbf{v}_2\cos(\omega_2 t + \phi_2)$$

Componente a componente:

$$\begin{aligned} \theta_a(t) &= A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) \\ \theta_b(t) &= A_1\cos(\omega_1 t + \phi_1) - A_2\cos(\omega_2 t + \phi_2) \end{aligned}$$

Ejemplo: batido

Condiciones iniciales: $\theta_a(0) = \theta_0$, $\theta_b(0) = 0$, $\dot{\theta}_a(0) = \dot{\theta}_b(0) = 0$.
Derivando y evaluando en $t = 0$:

$$\begin{aligned} \dot{\theta}_a(0) &= -A_1\,\omega_1\sin\phi_1 - A_2\,\omega_2\sin\phi_2 = 0 \\ \dot{\theta}_b(0) &= -A_1\,\omega_1\sin\phi_1 + A_2\,\omega_2\sin\phi_2 = 0 \end{aligned}$$

Sumando ambas:

$$-2A_1\,\omega_1\sin\phi_1 = 0 \quad \Longrightarrow \quad \phi_1 = 0$$

Restando:

$$-2A_2\,\omega_2\sin\phi_2 = 0 \quad \Longrightarrow \quad \phi_2 = 0$$

Con $\phi_1 = \phi_2 = 0$, las condiciones de posición quedan:

$$A_1 + A_2 = \theta_0 \qquad A_1 - A_2 = 0$$

De la segunda: $A_1 = A_2 \equiv A$. Sustituyendo en la primera:

$$2A = \theta_0 \quad \Longrightarrow \quad \boxed{A_1 = A_2 = \frac{\theta_0}{2}, \qquad \phi_1 = \phi_2 = 0}$$

Sustituyendo:

$$\theta_a(t) = \frac{\theta_0}{2}[\cos(\omega_1 t) + \cos(\omega_2 t)]$$
$$\theta_b(t) = \frac{\theta_0}{2}[\cos(\omega_1 t) - \cos(\omega_2 t)]$$

Usando $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$:

$$\theta_a(t) = \frac{\theta_0}{2}[\cos(\omega_1 t) + \cos(\omega_2 t)] = \frac{\theta_0}{2} \cdot 2\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)$$
$$= \theta_0 \cos\left(\frac{\omega_2 + \omega_1}{2}t\right)\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)$$

Usando $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$:

$$\theta_b(t) = \frac{\theta_0}{2}[\cos(\omega_1 t) - \cos(\omega_2 t)] = \frac{\theta_0}{2} \cdot 2\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\sin\left(\frac{\omega_2 - \omega_1}{2}t\right)$$
$$= \theta_0 \sin\left(\frac{\omega_2 + \omega_1}{2}t\right)\sin\left(\frac{\omega_2 - \omega_1}{2}t\right)$$

Sistema desacoplado y forzado

Planteo matricial

En forma matricial $\ddot{\mathbf{\Psi}} + \mathbf{D}\,\mathbf{\Psi} = 0$:

$$\mathbf{\Psi} = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} \frac{g}{\ell} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} \end{pmatrix}$$

Con autovalores y autovectores:

$$\omega_1^2 = \frac{g}{\ell}, \qquad \omega_2^2 = \frac{g}{\ell} + \frac{2k}{m}$$
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Desacople

Definimos $\mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, con $\mathbf{\Psi} = \mathbf{V}\,\mathbf{\Phi}$, $\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{\Phi} = \mathbf{V}^{-1}\mathbf{\Psi}$.

Partimos de $\ddot{\boldsymbol{\Psi}} = -\mathbf{D}\,\boldsymbol{\Psi}$. Multiplicamos a izquierda por $\mathbf{V}^{-1}$:

$$\mathbf{V}^{-1}\ddot{\boldsymbol{\Psi}} + \mathbf{V}^{-1}\mathbf{D}\,\boldsymbol{\Psi} = 0$$

Insertamos $\mathbf{V}\mathbf{V}^{-1} = \mathbf{I}$ entre $\mathbf{D}$ y $\boldsymbol{\Psi}$:

$$\mathbf{V}^{-1}\ddot{\boldsymbol{\Psi}} + (\mathbf{V}^{-1}\mathbf{D}\,\mathbf{V})\,\mathbf{V}^{-1}\,\boldsymbol{\Psi} = 0$$

$\mathbf{V}^{-1}\mathbf{D}\,\mathbf{V}$ es la diagonalización de $\mathbf{D}$:

$$\mathbf{V}^{-1}\mathbf{D}\,\mathbf{V} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

Usando $\boldsymbol{\Phi} = \mathbf{V}^{-1}\boldsymbol{\Psi}$:

$$\ddot{\boldsymbol{\Phi}} + \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\boldsymbol{\Phi} = 0 \implies \begin{cases} \ddot{\phi}_1 + \omega_1^2\,\phi_1 = 0, \\ \ddot{\phi}_2 + \omega_2^2\,\phi_2 = 0 \end{cases}$$

$$\implies \begin{cases} \phi_1(t) = A_1\cos(\omega_1 t + \varphi_1) \\ \phi_2(t) = A_2\cos(\omega_2 t + \varphi_2) \end{cases}$$

Forzado y amortiguado

$$\begin{cases} \ddot{\psi}_a + \Gamma\dot{\psi}_a + \dfrac{g}{\ell}\psi_a + \dfrac{k}{m}(\psi_a - \psi_b) = 0 \\ \ddot{\psi}_b + \Gamma\dot{\psi}_b + \dfrac{g}{\ell}\psi_b + \dfrac{k}{m}(\psi_b - \psi_a) = \dfrac{F_0}{m}\cos(\Omega t) \end{cases}$$

En forma matricial $\ddot{\boldsymbol{\Psi}} + \Gamma\dot{\boldsymbol{\Psi}} + \mathbf{D}\,\boldsymbol{\Psi} = \mathbf{F}$, con $\mathbf{F} = \begin{pmatrix} 0 \\ \frac{F_0}{m}\cos(\Omega t) \end{pmatrix}$.

Multiplicamos a izquierda por $\mathbf{V}^{-1}$:

$$\mathbf{V}^{-1}\ddot{\boldsymbol{\Psi}} + \Gamma\mathbf{V}^{-1}\dot{\boldsymbol{\Psi}} + \mathbf{V}^{-1}\mathbf{D}\,\boldsymbol{\Psi} = \mathbf{V}^{-1}\mathbf{F}$$

Insertamos $\mathbf{V}\mathbf{V}^{-1} = \mathbf{I}$ entre $\mathbf{D}$ y $\boldsymbol{\Psi}$:

$$\mathbf{V}^{-1}\ddot{\boldsymbol{\Psi}} + \Gamma\mathbf{V}^{-1}\dot{\boldsymbol{\Psi}} + (\mathbf{V}^{-1}\mathbf{D}\,\mathbf{V})\,\mathbf{V}^{-1}\,\boldsymbol{\Psi} = \mathbf{V}^{-1}\mathbf{F}$$

Usando $\boldsymbol{\Phi} = \mathbf{V}^{-1}\boldsymbol{\Psi}$:

$$\ddot{\boldsymbol{\Phi}} + \Gamma\dot{\boldsymbol{\Phi}} + \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\boldsymbol{\Phi} = \mathbf{V}^{-1}\mathbf{F}$$

Con $\mathbf{V}^{-1} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

$$\mathbf{V}^{-1}\mathbf{F} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{F_0}{m}\cos(\Omega t) \end{pmatrix} = \begin{pmatrix} \dfrac{F_0}{2m}\cos(\Omega t) \\ -\dfrac{F_0}{2m}\cos(\Omega t) \end{pmatrix}$$

Queda:

$$\begin{cases} \ddot{\phi}_1 + \Gamma\dot{\phi}_1 + \omega_1^2\,\phi_1 = \dfrac{F_0}{2m}\cos(\Omega t) = \dfrac{F_1}{m}\cos(\Omega t) \\ \ddot{\phi}_2 + \Gamma\dot{\phi}_2 + \omega_2^2\,\phi_2 = -\dfrac{F_0}{2m}\cos(\Omega t) = \dfrac{F_2}{m}\cos(\Omega t) \end{cases}$$

Solución

$$\begin{cases} \phi_1(t) = C_1 e^{-\Gamma t/2} \cos(\tilde{\omega}_1 t + \varphi_1) + A_1 \sin(\Omega t) + E_1 \cos(\Omega t) \\ \phi_2(t) = A_2 e^{-\Gamma t/2} \cos(\tilde{\omega}_2 t + \varphi_2) + A_2 \sin(\Omega t) + E_2 \cos(\Omega t) \end{cases}$$

con $\tilde{\omega}_i = \sqrt{\omega_i^2 - \Gamma^2/4}$, y:

$$A_i = \frac{\Gamma\Omega\,(F_i/m)}{(\omega_i^2 - \Omega^2)^2 + \Gamma^2\Omega^2}, \qquad E_i = \frac{(\omega_i^2 - \Omega^2)\,(F_i/m)}{(\omega_i^2 - \Omega^2)^2 + \Gamma^2\Omega^2}$$